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An Effective Numerical Method for Linear Mode Conversion Problems

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Linear mode conversion, an important issue in the physics of plasma waves, involves an ordinary differential equation of at least fourth order. Attempts to integrate this differential equation using conventional numerical methods typically fail to provide a physically sensible result. This failure occurs because small, unavoidable truncation and differencing errors excite nonphysical, mathematically allowed exponentially growing solutions which quickly overwhelm the desired physical solution. We present here a two-point boundary numerical method which avoids exciting these unwanted nonphysical solutions and so provides solutions that are physically significant. The numerical algorithm is a generalization of the standard tridiagonal method and provides single-pass solutions which satisfy the original difference equations to within the numerical accuracy of the computer.

1. INTRODUCTION

Many different waves can propagate in a plasma. When the plasma is spatially uniform, qualitatively different waves are linearly independent. However, when the plasma is spatially nonuniform, there can be strong localized coupling between a pair of qualitatively different modes. This coupling typically occurs when the wavelengths of the two modes depend on plasma position and at some critical location have the same value. One mode may convert completely into the other at this location, or there may be some combination of partial transmission, partial reflection, and partial conversion. Mathematically, linear mode conversion is a boundary layer phenomenon and is associated with a fourth or higher order ordinary differential equation (ODE). Many linear mode conversion situations can be expressed in terms of the "standard equation" [1–4]

$$\frac{d^4E}{dx^4} + x\frac{d^2E}{dx^2} + \Gamma E = 0, \tag{1}$$

which has been arranged to have mode conversion at $x = 0$.

For purposes of comparison with the numerical technique presented later, we briefly summarize the standard analytic method used to obtain asymptotic solutions of Eq. (1): First, the equation is Laplace transformed to obtain a first-order ODE in the variable $E(p)$. The first-order ODE is easily solved for $E(p)$, and then $E(x)$ is found from the Bromwich integral of $E(p)$, i.e. from a contour integral in the complex p plane. Because Eq. (1) is fourth-order, there are four distinct integration paths for the contour integral, each giving one of the four independent solutions to Eq. (1). The appropriate linear combination of solutions is found by imposing four physically relevant boundary conditions. This turns out to be equivalent to assigning the ends of the integration paths, since analyticity within a region means that a given integration path with fixed end points can be deformed in that region and still give the same result. The end points are typically determined using boundary conditions at $x \to \pm\infty$; these boundary conditions at infinity are expressed in terms of qualitative properties of the modes, e.g., "no slow wave propagates to the right," or "all modes are bounded at infinity." By deforming the integration paths through saddle points in the p-plane, steepest descent methods can be used to obtain large $|x|$ asymptotic solutions and, hence, matching formulae can be obtained for WKB (Wentzl–Kramers–Brillouin) solutions valid outside the neighborhood of $x = 0$. This analytic procedure is intricate and may not work for systems more complicated than Eq. (1).

Equation (1) is usually a local approximation of some more complicated equation, and, if quantitative results are required, it would seem reasonable to attempt a numerical integration of the original equation. It is tempting to try to solve this equation using standard numerical methods (e.g., Runge–Kutta). However, it is found that mathematically allowed, nonphysical exponentially growing solutions overwhelm the physical solution no matter which way the solution proceeds from the $x = 0$ layer. This failure occurs because small numerical errors (e.g., truncation or differencing errors) always introduce a small "seed" of exponentially growing wave, even if the unwanted mode had not been imposed as part of the boundary conditions. These exponentially growing modes quickly dwarf the desired physically relevant solution giving a nonsensical result.

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The analytic method outlined earlier explicitly casts out unwanted modes by imposing boundary conditions at two points, one on each side of the mode conversion layer. This suggests that the unwanted, nonphysical exponentially growing modes might be avoided in a numerical solution by imposing boundary conditions at two points [5], one on each side of the boundary layer.

This problem has been dealt with by various methods in previous work. In general, these approaches impose two-point boundary conditions either by matching to analytical solutions on either side of the conversion layer or by using implicit numerical schemes. The analytical matching approach was used by Colestock and Kashuba [6] and by Romero and Scharer [7] who solved the equations numerically over a narrow range and then matched the numerical solutions to WKB analytical solutions satisfying the imposed boundary conditions on both sides. Colestock and Kashuba remarked on the need to ensure that the numerical domain is wide enough so the WKB solutions are good approximations, but not so wide that roundoff error would cause rapid exponential growth of the unwanted, nonphysical modes. Jaegar, Batchelor, and Weitzner [8] used a fully implicit global numerical scheme to solve sixth-order equations in one step, but did not give details of their method for inverting the difference equations. Ross, Chen, and Mahajan [9] expanded the dependent variables in terms of cubic spline functions, then applied a projection operator incorporating two-point boundary conditions to the differential equation, and so obtained an $N \times N$ matrix where N was the number of grid points. They inverted the matrix using Gaussian elimination to determine the coefficients of the cubic spline functions and so provide a solution to the global problem.

We present here a new and very straightforward single-pass implicit numerical scheme for solving the two-point boundary problem. Because this scheme is not computationally intensive, an ordinary desktop computer is more than adequate for solving complicated, high resolution, realistic problems.

The basic problem to be solved is the system of two second-order ODEs coupled by terms linear in the dependent variables:

$$\begin{aligned} f_1(x)\psi'' + g_1(x)\psi' + h_1(x)\psi &= \lambda(x)\chi \\ f_2(x)\chi'' + g_2(x)\chi' + h_2(x)\chi &= \mu(x)\psi. \end{aligned} \tag{2}$$

Equation (2) is a generalization of the standard equation, Eq. (1), since if one sets $d^2E/dx^2 = \psi$, Eq. (1) can be written

$$\begin{aligned} \psi'' + x\psi &= -\Gamma E \\ E'' &= \psi. \end{aligned} \tag{3}$$

The paper is organized as follows: Section 2 presents the derivation of the algorithm for solving Eqs. (2). Section 3 generalizes this method to solve systems of M coupled equations in M dependent variables. Section 4 discusses examples for some actual mode conversion problems. Section 5 contains a brief summary and conclusions.

2. TRIDIAGONAL SCHEME FOR TWO COUPLED SECOND-ORDER ODES

We now derive the algorithm for solving Eq. (2); this algorithm is a generalization of the tridiagonal matrix inversion scheme commonly used for solving an individual second-order ODE having boundary conditions at two different points. By expressing the derivatives in finite difference form, i.e.,

$$\psi' = \frac{\psi_{i+1} - \psi_{i-1}}{2\Delta}, \quad \psi'' = \frac{\psi_{i+1} + \psi_{i-1} - 2\psi_i}{\Delta^2},$$

Eq. (2) becomes the pair of coupled difference equations,

$$\begin{aligned} A_i^- \psi_{i-1} + A_i^0 \psi_i + A_i^+ \psi_{i+1} &= \lambda_i \chi_i, \\ B_i^- \chi_{i-1} + B_i^0 \chi_i + B_i^+ \chi_{i+1} &= \mu_i \psi_i, \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_i^- &= \frac{f_{1i}}{\Delta^2} - \frac{g_{1i}}{2\Delta}, \\ A_i^0 &= -\frac{2f_{1i}}{\Delta^2} + h_{1i}, \\ A_i^+ &= \frac{f_{1i}}{\Delta^2} + \frac{g_{1i}}{2\Delta}, \end{aligned} \tag{5}$$

and similarly for the B coefficients.

To solve Eq. (4), we assume that ψ and χ at location $i + 1$ are *linear* functions of ψ and χ at location i; i.e., we assume there exists the relationship

$$\begin{aligned} \psi_{i+1} &= \alpha_i \psi_i + \beta_i \chi_i + \varepsilon_i, \\ \chi_{i+1} &= \gamma_i \psi_i + \kappa_i \psi_i + \delta_i, \end{aligned} \tag{6}$$

where the coefficients $\alpha_i, \beta_i, \varepsilon_i, \gamma_i, \kappa_i$, and ε_i are to be determined.

To find these coefficients, we use Eq. (6) to eliminate ψ_{i+1} and χ_{i+1} in Eq. (4) and then solve the resulting equations for ψ_i and χ_i to obtain

$$\begin{aligned} \psi_i &= a_i \chi_i + b_i \psi_{i-1} + c_i, \\ \chi_i &= d_i \psi_i + e_i \chi_{i-1} + f_i, \end{aligned} \tag{7}$$

where

$$a_i = \frac{\lambda_i - \beta_i A_i^+}{A_i^0 + \alpha_i A_i^+}, \quad b_i = -\frac{A_i^-}{A_i^0 + \alpha_i A_i^+}, \quad c_i = -\frac{\varepsilon_i A_i^+}{A_i^0 + \alpha_i A_i^+},$$
$$d_i = \frac{\mu_i - \kappa_i B_i^+}{B_i^0 + \gamma_i B_i^+}, \quad e_i = -\frac{B_i^-}{B_i^0 + \gamma_i B_i^+}, \quad f_i = -\frac{\delta_i B_i^+}{B_i^0 + \gamma_i B_i^+}. \tag{8}$$

Equation (7) is then solved for ψ_i and χ_i in terms of ψ_{i-1} and χ_{i-1}, giving

$$\psi_i = \frac{b_i}{1 - a_i d_i}\psi_{i-1} + \frac{a_i e_i}{1 - a_i d_i}\chi_{i-1} + \frac{c_i + a_i f_i}{1 - a_i d_i},$$
$$\chi_i = \frac{e_i}{1 - a_i d_i}\chi_{i-1} + \frac{d_i b_i}{1 - a_i d_i}\psi_{i-1} + \frac{f_i + d_i c_i}{1 - a_i d_i}. \tag{9}$$

By replacing the dummy index i with $i - 1$, Eq. (6) can be rewritten as

$$\psi_i = \alpha_{i-1}\psi_{i-1} + \beta_{i-1}\chi_{i-1} + \varepsilon_{i-1},$$
$$\chi_i = \gamma_{i-1}\chi_{i-1} + \kappa_{i-1}\psi_{i-1} + \delta_{i-1}. \tag{10}$$

Comparison of Eqs. (9) and (10) shows that the recursion relations for the sought-after coefficients are

$$\alpha_{i-1} = \frac{b_i}{1 - a_i d_i}, \quad \beta_{i-1} = \frac{a_i e_i}{1 - a_i d_i}, \quad \varepsilon_{i-1} = \frac{c_i + a_i f_i}{1 - a_i d_i},$$
$$\gamma_{i-1} = \frac{e_i}{1 - a_i d_i}, \quad \kappa_{i-1} = \frac{d_i b_i}{1 - a_i d_i}, \quad \delta_{i-1} = \frac{f_i + d_i c_i}{1 - a_i d_i}. \tag{11}$$

The actual coefficients are determined using a descending iteration scheme (analogous to the usual tridiagonal method) as follows: Since ψ_N and χ_N are specified, setting $i = N$ in Eq. (10) gives

$$\alpha_{N-1} = \beta_{N-1} = \gamma_{N-1} = \kappa_{N-1} = 0 \tag{12}$$

and

$$\varepsilon_{N-1} = \psi_N, \quad \delta_{N-1} = \chi_N. \tag{13}$$

Knowing $\alpha_{N-1}, \beta_{N-1}, \gamma_{N-1}, \kappa_{N-1}, \varepsilon_{N-1}$, and δ_{N-1} allows $a_{N-1} \cdots f_{N-1}$ to be calculated using Eq. (8), and so α_{N-2} etc. can be determined using Eq. (11). Repeating this recursive procedure determines $\{\alpha_i, \beta_i, \varepsilon_i, \gamma_i, \kappa_i, \delta_i\}$ for the entire range $i = 0$ to $N - 1$.

Using the boundary conditions ψ_0 and χ_0 and Eq. (6) allows ψ_i and χ_i to be calculated for $1 \le i \le N - 1$, thus giving the desired solution of Eq. (2).

This scheme is an exact solution of the coupled difference equations. Because of the speed and accuracy of the scheme, it is feasible to use a very fine numerical grid (i.e., large N) to obtain high resolution.

3. EXTENSION TO ARBITRARY NUMBER OF COUPLED SECOND-ORDER ODES

Although Eq. (2) is quite general and applies to many problems, there exist situations involving more than two dependent variables. For example, one could have a situation with three dependent variables having the coupled equations

$$f_1(x)\psi'' + g_1(x)\psi' + h_1(x)\psi = \lambda(x)\chi + \sigma(x)\phi$$
$$f_2(x)\chi'' + g_2(x)\chi' + h_2(x)\chi = \mu(x)\psi + \rho(x)\phi \tag{14}$$
$$f_3(x)\phi'' + g_3(x)\phi' + h_3(x)\phi = \theta(x)\psi + \tau(x)\chi.$$

Rather than working out the extension to three variables, which is tedious, we now derive the general extension to M dependent variables, $\{\psi^{(1)}, \psi^{(2)}, ..., \psi^{(M)}\}$. Thus, we consider the set of M coupled second-order ODEs,

$$f^{(j)}(x)\frac{d^2\psi^{(j)}}{dx^2} + g^{(j)}(x)\frac{d\psi^{(j)}}{dx} + h^{(j)}(x)\psi^{(j)} = \sum_{k \ne j} \lambda^{(j,k)}(x)\psi^{(k)}, \quad j = 1, M. \tag{15}$$

These equations can be put into finite difference form as

$$D_i^{(j)}\psi_{i-1}^{(j)} + S_i^{(j)}\psi_i^{(j)} + U_i^{(j)}\psi_{i+1}^{(j)} = \sum_{k \ne j} \lambda_i^{(j,k)}\psi_i^{(k)}, \tag{16}$$

where the down (D), same (S), and up (U) coefficients are, in analogy to Eq. (5),

$$D_i^{(j)} = \frac{f_i^{(j)}}{\Delta^2} - \frac{g_i^{(j)}}{2\Delta}$$
$$S_i^{(j)} = -\frac{2f_i^{(j)}}{\Delta^2} + h_i^{(j)} \tag{17}$$
$$U_i^{(j)} = \frac{f_i^{(j)}}{\Delta^2} + \frac{g_i^{(j)}}{2\Delta}.$$

We define the diagonal element

$$\lambda_i^{(j,j)} = -S_i^{(j)}, \tag{18}$$

so that Eq. (16) can be written as

$$D_i^{(j)}\psi_{i-1}^{(j)} + U_i^{(j)}\psi_{i+1}^{(j)} = \sum_{k=1}^{M} \lambda_i^{(j,k)}\psi_i^{(k)}. \tag{19}$$

In analogy to Eq. (6), we postulate that the solution at location $i + 1$ is a linear function of the solutions at the location i and write

$$\psi_{i+1}^{(j)} = \varepsilon_i^{(j)} + \sum_{k=1}^{M} T_i^{(j,k)} \psi_i^{(k)}, \tag{20}$$

where $\varepsilon_i^{(j)}$ and $T_i^{(j,k)}$ are coefficients to be determined.

3.1. *Determination of* $\varepsilon_i^{(j)}$ *and* $T_i^{(j,k)}$

Using Eq. (20) to substitute for $\psi_{i+1}^{(j)}$ in Eq. (19), we obtain

$$D_i^{(j)} \psi_{i-1}^{(j)} + U_i^{(j)} \left[\varepsilon_i^{(j)} + \sum_{k=1}^{M} T_i^{(j,k)} \psi_i^{(k)} \right] = \sum_{k=1}^{M} \lambda_i^{(j,k)} \psi_i^{(k)}, \tag{21}$$

or, after modest rearrangement,

$$\sum_{k=1}^{M} Q_i^{(j,k)} \psi_i^{(k)} = D_i^{(j)} \psi_{i-1}^{(j)} + U_i^{(j)} \varepsilon_i^{(j)}, \tag{22}$$

where

$$Q_i^{(j,k)} = \lambda_i^{(j,k)} - U_i^{(j)} T_i^{(j,k)}. \tag{23}$$

We define $P_i^{(l,n)}$ to be the inverse of $Q_i^{(j,k)}$; i.e.,

$$\sum_{j=1}^{M} P_i^{(l,j)} Q_i^{(j,k)} = \delta_{lk}, \tag{24}$$

premultiply Eq. (22) by $P_i^{(l,j)}$, sum over j, let $l \to j$, change the summation index to be k, and obtain

$$\psi_i^{(j)} = \sum_{k=1}^{M} P_i^{(j,k)} U_i^{(k)} \varepsilon_i^{(k)} + \sum_{k=1}^{M} P_i^{(j,k)} D_i^{(k)} \psi_{i-1}^{(k)}. \tag{25}$$

However, letting $i \to i - 1$ in Eq. (20) gives

$$\psi_i^{(j)} = \varepsilon_{i-1}^{(j)} + \sum_{k=1}^{M} T_{i-1}^{(j,k)} \psi_{i-1}^{(k)}. \tag{26}$$

Comparing Eqs. (25) and (26), we identify

$$T_{i-1}^{(j,k)} = P_i^{(j,k)} D_i^{(k)} \tag{27}$$

and

$$\varepsilon_{i-1}^{(j)} = \sum_{k=1}^{M} P_i^{(j,k)} U_i^{(k)} \varepsilon_i^{(k)}. \tag{28}$$

Thus, the sought-after coefficients are determined in a backwards recursion scheme that is an extension of the tridiagonal method. Note that the $P_i^{(j,k)}$ and $Q_i^{(j,k)}$ are used only one i at a time and so may be stored as two temporary $M \times M$ matrices that are overwritten for each new i.

3.2. *Backwards Recursion Scheme*

Since the $\psi_N^{(j)}$ are specified, setting $i = N$ in Eq. (26) gives

$$\varepsilon_{N-1}^{(j)} = \psi_N^{(j)} \tag{29}$$

and

$$T_{N-1}^{(j,k)} = 0. \tag{30}$$

Knowing $T_{N-1}^{(j,k)}$ allows $Q_{N-1}^{(j,k)}$ to be calculated using Eq. (23). Knowing $Q_{N-1}^{(j,k)}$ allows the inverse $P_{N-1}^{(j,k)}$ to be found.

The process is then repeated; i.e., knowing $P_{N-1}^{(j,k)}$ and $\varepsilon_{M-1}^{(j)}$ allows $T_{N-2}^{(j,k)}$ and $\varepsilon_{N-2}^{(j)}$ to be calculated using Eqs. (27) and (28). The process is then continued down until $T_0^{(j,k)}$ and $\varepsilon_0^{(j)}$ have been found.

Knowing $\varepsilon_i^{(j)}$ and $T_i^{(j,k)}$ for $0 \le i \le N - 1$ allows determining all $\psi_i^{(j)}$ in the range $1 \le i \le N - 1$; recall that $\psi_0^{(j)}$ and $\psi_N^{(j)}$ were specified as the two-point boundary conditions.

The only computationally intensive part of this scheme is the inversion of the $M \times M$ matrix $Q_i^{(j,k)}$ which has to be done at each value of the independent variable, i.e. $N - 1$ times. Thus, the complete numerical integration involves $\mathcal{O}(M^3 N)$ computations.

4. SOME EXAMPLES

4.1. *The Standard Equation*

The WKB-like Fourier transform of Eq. (1), valid away from $x = 0$, is

$$k^4 - xk^2 + \Gamma = 0 \tag{31}$$

which has roots

$$k^2 = \frac{x \pm \sqrt{x^2 - 4\Gamma}}{2}. \tag{32}$$

For $|x| \gg 2\sqrt{\Gamma}$, the two roots are distinct and consist of a large root,

$$k_l^2 = x, \tag{33}$$

and a small root,

$$k_s^2 = \Gamma / x. \tag{34}$$

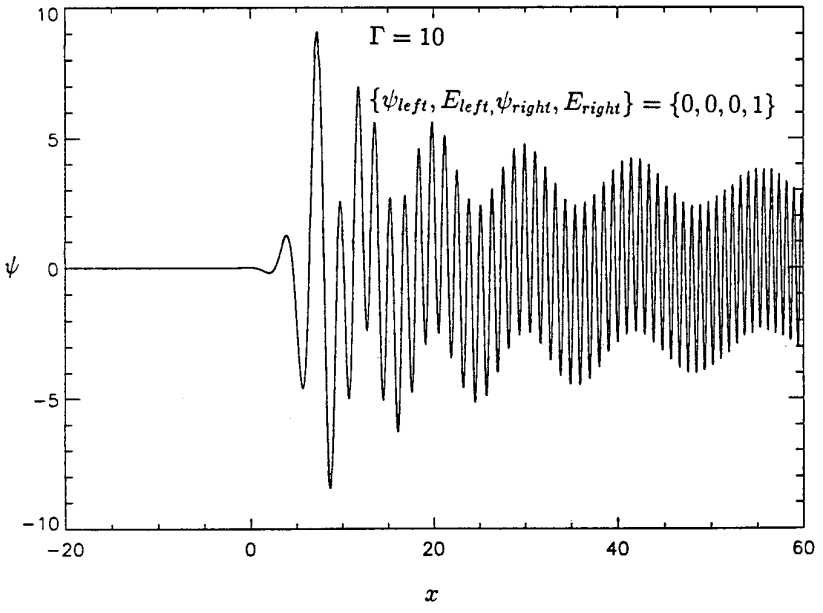

FIG. 1. Numerical solution of Eq. (3) for $\Gamma = 10$ on the domain $-20 \le x \le 60$ using a 1000-point grid and the boundary condition $\{\psi_{\text{left}}, E_{\text{left}}, \psi_{\text{right}}, E_{\text{right}}\} = \{0, 0, 0, 1\}$.

The large root (i.e., short wavelength mode) corresponds to balancing the first and middle terms in Eq. (31), while the small root corresponds to balancing the middle and last terms. If $\Gamma > 0$, then for $x \ll -1$ both of these roots give nonoscillatory exponentially growing or decaying solutions. When $x \simeq 2\sqrt{\Gamma}$ the two roots coalesce, but this is also the region where the WKB description is invalid.

Figure 1 shows solutions to Eq. (3) for $\Gamma = 10$ on the domain $-20 \le x \le 60$ using a 1000-point grid and the boundary condition $\{\psi_{\text{left}}, E_{\text{left}}, \psi_{\text{right}}, E_{\text{right}}\} = \{0, 0, 0, 1\}$. For $x > 20$, the long and short wavelength modes are clearly distinguishable and it is seen that the wavelength of the long wavelength mode increases with increasing x while the opposite is true for the short wavelength modes. The coalescence region is seen to be in the vicinity of the $x \simeq 2\sqrt{\Gamma} \approx 6$, as suggested by the approximate WKB model. The boundary condition $\{0, 0, 1, 0\}$ gives a solution that is essentially similar to the $\{0, 0, 0, 1\}$ case shown in Fig. 1, but is orthogonal in phase.

The two other independent solutions obtained using the respective boundary conditions $\{1, 0, 0, 0\}$ and $\{0, 1, 0, 0\}$ give solutions that decay exponentially with increasing x. Figure 2 shows the numerically calculated solution for the case of $\{0, 1, 0, 0\}$. For this case and also for $\{1, 0, 0, 0\}$ the boundary condition is forcing a mode to be finite at the left-hand side of the x domain, where the mode is evanescent. Since the right-hand boundary condition requires the mode to vanish, the mode decays exponentially with increasing x.

4.2. *Alfven Wave Mode Conversion*

In Eq. (3) it is clear that the coupling term is intrinsic to the existence of one of the modes; i.e., if Γ is set to zero then there is only one wave-like mode. A slightly different situation occurs when there are two coupled second-order equations which would still exist even if the coupling terms were set to zero. An example of this is the problem of mode conversion between fast (compressional) and slow (shear) cold Alfven waves. The equations governing this coupling for a radially inhomogeneous cyclindrically symmetric plasma are [10, 11]

$$\frac{1}{r}\frac{d}{dr}\left[\frac{rS}{S-n_z^2}\frac{d\tilde{E}_z}{dr}\right] + \left[-\frac{m^2 S}{r^2(S-n_z^2)} + P\right]\tilde{E}_z = -\frac{imn_z}{r}\tilde{B}_z\frac{d}{dr}\left(\frac{1}{S-n_z^2}\right) \tag{35}$$

and

$$\frac{1}{r}\frac{d}{dr}\left[\frac{r}{S-n_z^2}\frac{d\tilde{B}_z}{dr}\right] + \left[-\frac{m^2}{r^2(S-n_z^2)} + 1\right]\tilde{B}_z = \frac{imn_z}{r}\tilde{E}_z\frac{d}{dr}\left(\frac{1}{S-n_z^2}\right), \tag{36}$$

where m is the azimuthal mode number, $S(r)$ is the perpendicular plasma dielectric, $P(r)$ is the parallel plasma dielec-

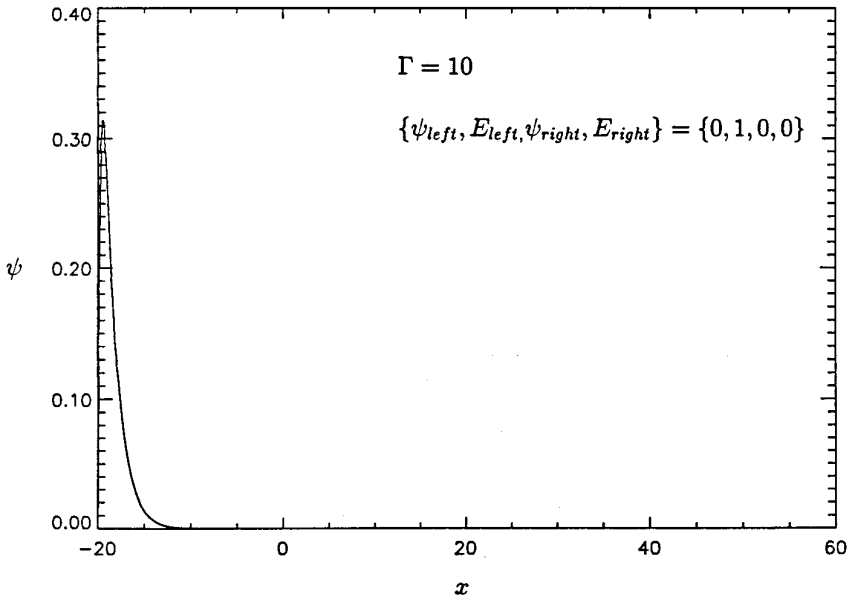

FIG. 2. Same as Fig. 1, except $\{\psi_{\text{left}}, E_{\text{left}}, \psi_{\text{right}}, E_{\text{right}}\} = \{0, 1, 0, 0\}$.

tric, and n_z is the parallel refractive index. Thus, if $m = 0$ or if the plasma is uniform, the two equations describe two decoupled modes, the $\tilde{E}_z$ mode (slow, shear wave) and the $\tilde{B}_z$ mode (fast, compressional mode). Typical numerical solutions for these equations using the method presented here are given for tokamak parameters in Figs. 3 and 4 of Ref. [10] and for magnetospheric parameters in Fig. 1 of Ref. [11]. It is interesting to note that the Laplace transform analytic method is of no use for Eqs. (35) and (36), since Laplace transforming these equations leads to a pair of coupled second-order ODEs in the Laplace transform variables which are no easier to solve than Eqs. (35) and (36).

5. SUMMARY AND CONCLUSIONS

We have presented a numerical scheme for solving mode conversion problems involving an arbitrary number of coupled second-order linear ODEs. This scheme is a generalization of the standard tridiagonal scheme for two-point boundary problems and is easy to implement. Examples have been provided for the standard equation of mode conversion and also for the more complicated problem of Alfven wave mode conversion. An example involving a system of four coupled ODEs has also been worked out, but the derivation of this system requires a lengthy discussion about the underlying physics, and so the presentation of this more complicated example will be deferred to a future physics-oriented paper.

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